Banking, Leaning and Conical Swings

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Banked corners

Highways are built with banked corners. (The outside edge on bends is higher than the inside edge.) The forces acting on a car at high speed in a banked corner are shown in figure 1.

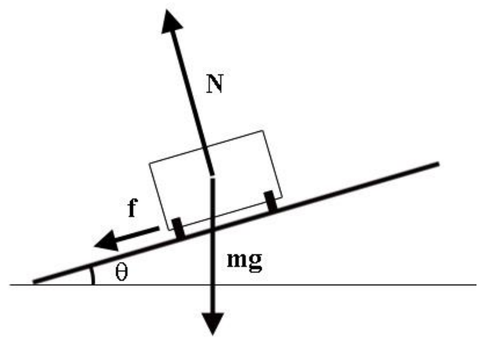

Fig 1 – three forces act on a car of mass *m* as it turns to the left in a banked corner.

The centripetal force (an unbalanced force acting towards the centre of the turning circle of radius r) is the sum of two forces. The horizontal component of the friction force, $f\cos\theta$, and the horizontal component of the normal force, $N\sin\theta$.

The ideal case

Figure 2 shows a car of mass m and speed v turning to the left. The speed has been reduced so that no inward friction force acts between the tires and the road. At this speed only two forces remain on the car (and the passengers).

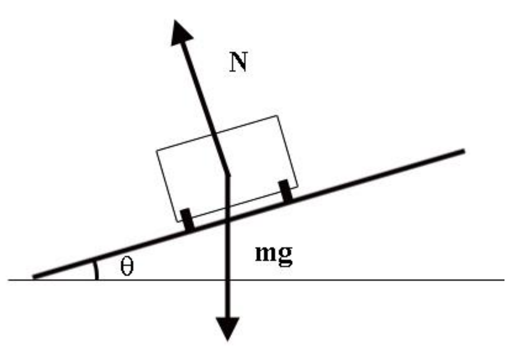

Fig 2 – there is no friction on the incline. The two remaining forces are shown.

The unbalanced centripetal force is now the horizontal component of the normal force *N*.

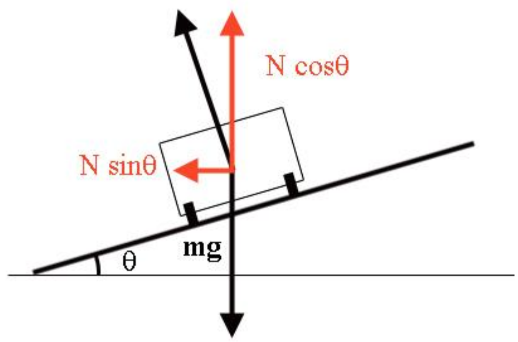

Fig 3 – the components of the normal force in red.

In the horizontal direction … $$N \sin \theta = \frac{mv^2}{r}$$

In the vertical direction … $$N \cos \theta = mg$$

Dividing the first equation by the second and cancelling *m* gives …

$$\tan \theta = \frac{v^2}{rg} \qquad \text{… [1]}$$

A speed *v* the optimum banking angle is given by …

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right) \qquad \text{… [2]}$$

Rearranging gives the optimum speed in a corner of radius *r* and banking angle θ as ...

$$v = \sqrt{rg \tan \theta} \qquad \text{… [3]}$$

Imagine a smooth level road (neither rising nor falling) with a centre line that curves right and left. The road surface rotates so that at every point in a turn equation [1] is exactly true. A glass of water, full to the brim, rests on a box between the front seats. The car is driven at constant speed.

Imagine that you sit up straight in the middle of the back seat and close your eyes. You cannot tell when the car turns a corner, but you do notice that your weight is very slightly increased at times. Open your eyes. Not a drop of water has been spilt. In a real car on a real road this may be nearly if not exactly true, but most times divers take corners too fast.

A passenger in the car (an accelerated frame of reference)

In a bend on a flat road a passenger feels as though an outward force pushes them towards the outer door. If the door burst open they would drop onto the road and bounce along in a straight line tangent to the bend at their point of exit. The apparent outward force, called an *inertial force*, is due to the acceleration of the frame of reference (the car). An inertial force only appears in an accelerated frame and is not real in this sense: but be careful, inertial forces are not *imaginary*, as anyone who has been in a car accident knows.

Figure 2 has been redrawn with *m* now being the mass of a passenger. The outward inertial force experienced by the passenger is in red.

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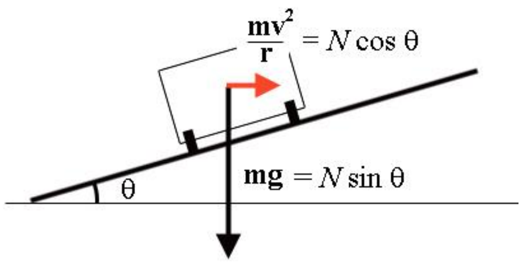


Fig 3 – a passenger feels two forces, their weight *mg,* and an outward inertial force (in red).

The inertial force adds to the effect of gravity, increases their weight, and rotates the vertical. To see by how much we must add the two vectors.

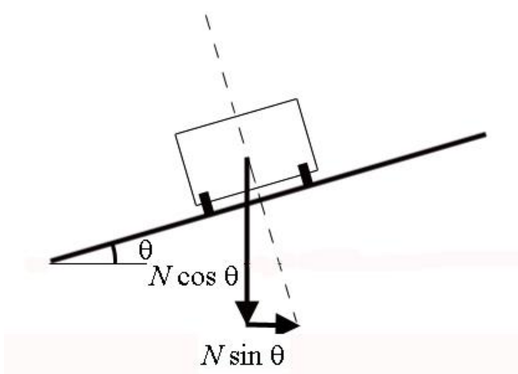


Fig 4 – the rotation of the vertical.

Writing the lengths of the vectors as $N\cos\theta$ and $N\sin\theta$ shows at once that the vertical is rotated by an angle θ and remains perpendicular to the banked road. In this special case, at just the right speed in the banked corner, a passenger can safely drink a cup of coffee.

Lean angle

When turning a corner on a bike you must lean. The most *comfortable* angle rotates the bike as though the corner were banked and the vertical is aligned with the bike and the rider.

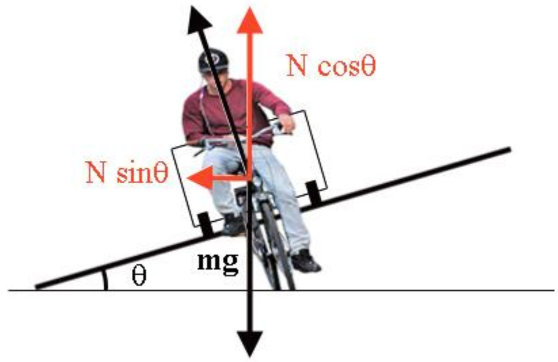


Fig 7 – taking a flat corner at speed *v*.

The optimum angle is given by equation [1] as …

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$$

Note: a leaning bike is unstable in this simple model, but stability in corners depends on other factors as it does when riding upright in a straight line. t has been found that and no single factor is responsible and the exact details are the subject of ongoing discussion.

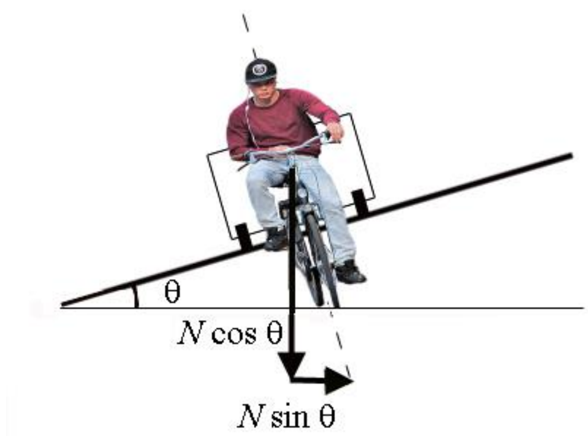


Fig 8 – figure 4 has been redrawn.

The angle of lean is aligned with the vertical as it is for the optimum banking angle.

The coefficient of friction (in the limiting case)

For a bike in a flat corner the centripetal force is the inward friction force between the tires and the road. The coefficient of rolling friction along a radius is a little less than the in-line coefficient of friction that keeps the bike driving at constant speed against air resistance. In the limiting case, just before the bike slides out, we can write …

$$\frac{mv^2}{r} = \mu mg$$

… where μ is the coefficient of .static rolling friction.

Substituting from equation [1] for v^2 …

$$\mu = \tan\theta$$

…. where θ is the optimum lean angle just before sliding occurs.

A demonstration

It would be possible in principle to demonstrate the effect of optimum velocity in a banked corner with a glass of water on a cart on rails but equation [1] cannot easily be satisfied exactly. Rotation of the vertical in a flat corner can be demonstrated by holding a glass of water at arms-length and swinging it in a circle but another way must be found to demonstrate the condition for optimum banking angle at speed *v*.

Figure 9 shows the forces acting on a small bucket of water swinging on a cord in a circle.

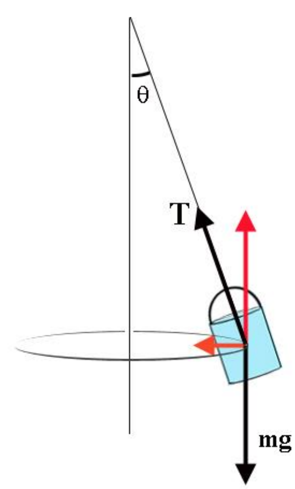


Fig 9 – a bucket of water swings in a circle.

The two forces and their components in figure 9 are identical with those in figure 3. Equation [1] applies in this situation, which is easily demonstrated by swinging a bucket of water by hand. As the velocity is increased the angle θ increases and the vertical rotates to keep the water surface flush with the rim.

Questions

1 Suppose a 180 degree semicircular bend on a highway has a radius of 250 m and the suggested optimum speed is 25 m/s. What is the optimum banking angle?

2 The positive banking angle of a tight circular corner (r =100 m) on a race track is 30°.

a Find the speed that would align the vertical in the corner with the vertical axis of the car and write down the centripetal acceleration in the corner at this constant speed.

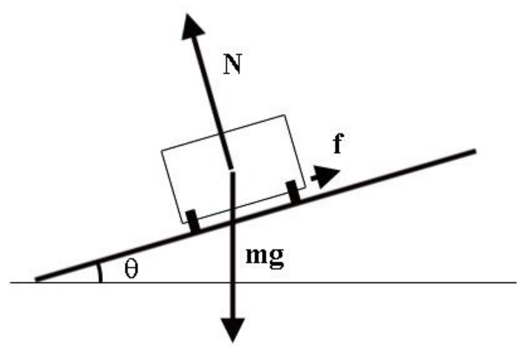
b Sketch the vector diagram of forces on the car at twice the optimum speed.

c Discuss the effect on a passenger in the car turning the corner at twice the optimum speed and outline the physical reasons for his experience.

3 Friction acts up the plane.

a Write down an expression for the centripetal force on the car in the corner in terms of N, θ and f.

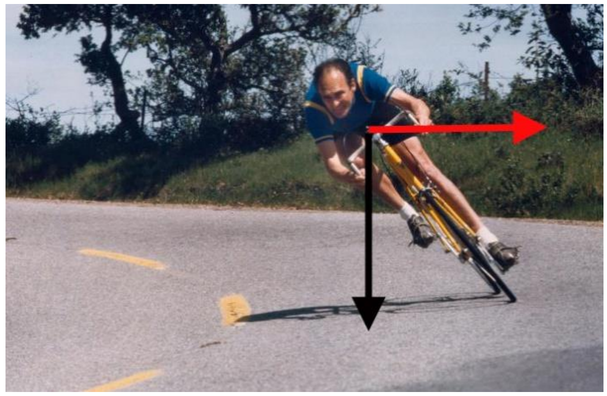
b* Discuss the experience of a passenger in the car at this low speed.



5* Corners with positive camber are higher on the outside than the inside. An *off-camber corner* is the opposite: lower on the outside than the inside. Off-camber corners are feared by average drivers. Discuss the physics.

6 The steering linkage is unusual. The handlebars are not rotated but are leaned left and right. Has the designer got the left/right directions right? Briefly explain your reasoning.

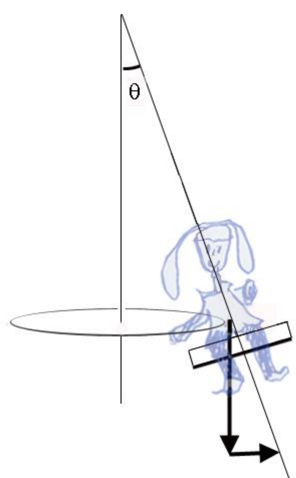
4 Weight (black) and inertial force (red) are shown on the diagram to scale.



a How closely is the rider aligned with the optimum lean angle?
(*Take measurements and add vectors*.)

b If the bike is about to slide out of the corner find the coefficient of static rolling friction between the tires and the road.

6 A child is shown on a conical swing. The tension in the light rope is T, the seat and child have a combined mass m and the circular path has radius r.



a Identify the vectors and show that their magnitudes are given by …

$$T\cos\theta = mg \quad \text{and} \quad T\sin\theta = \frac{mv^2}{r}$$

b* Children perfer to play on in-line swings. When seats are hung on chaines on merry-go-rounds in fair grounds, ride times are short, and the music is loud and exciting.

Discuss the physics: why are in-line swings more interesting?