Bernoulli's equation

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Conservation of energy in fluid flow

As a mass *m* falls a distance *h*, work is done by gravity and potential energy becomes kinetic energy. Neglecting air resistance …

$$mgh = \tfrac{1}{2}mv^2$$

To apply this idea (*the conservation of energy*) to flowing air or water requires a little thought, but the result (due originally to Daniel Bernoulli and still carrying his name) is well known. It is hoped that the derivation below has been set out in sufficient detail to allow the reader to carefully follow each step.

Imagine that a steady flow of water is pumped to the right from level 1 to level 2 (figure 1). In the real world drag opposes the motion because water has viscosity (is a little bit like honey) and pipes often have sharp bends and sudden changes in radius that generate eddies (turbulence) that converts linear kinetic energy to rotational energy. In the real world drag and turbulence remove some energy from the flow but here we wll neglect these small complications.

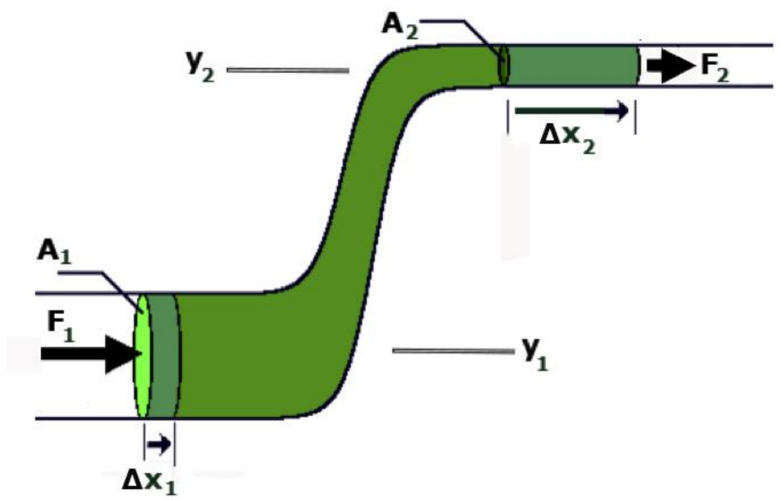

Fig 1 – Water is forced to the right. The flow is slow and the pipe smooth so we can neglect any energy loss as heat due to turbulence and viscous drag.

The pipe is long and filled with water. Consider the section of water (green), which we define as the 'system'. Work is done *on* the system at level y_1 and *by* the system at level y_2. F_1 does work $F_1\Delta x_1$ and F_2 does work $F_2\Delta x_2$.

The work done as the *system* is forced to the right is …

$$F_1\Delta x_1 - F_2\Delta x_2$$

Remembering that $F_1 = P_1A_1$ and $F_2 = P_2A_2$

… and $A_1\Delta x_1 = A_2\Delta x_2 = \Delta V$

The work done $(F_1\Delta x_1 - F_2\Delta x_2) = (P_1 - P_2)\Delta V$ … where P_1 and P_2 are the pressures in the fluid at level 1 and level 2.

Over time the element of volume $A_1\Delta x_1$ replaces $A_2\Delta x_2$ on the upper level. If kinetic energy is not converted to heat the work done on this element of volume must equal the sum of the change in kinetic energy and potential energy.

$$(P_1 - P_2)\Delta V = ½\Delta m v_1^2 - ½\Delta m v_2^2 + \Delta m g y_1 - \Delta m g y_2$$

If the fluid density is the same everywhere $\Delta m/\Delta V = \rho$

Rearranging gives ... $P_2 + ½\rho v_2^2 + \rho g y_2 = P_1 + ½\rho v_1^2 + \rho g y_1$ … [1]

Equation 1 is a simple form of Bernoulli's equation. Each term has the units of energy density (Joules per cubic metre). In particular note that the energy per cubic metre of air at sea level due to its compression is close to 10^5 J.

If we know all the conditions at location 1, and two conditions at location 2, we can find the one unknown.

Note: Bernoulli's equation applies to laminar flow without linear kinetic energy loss due to turbulence (chaotic rotations), in a fluid of constant density (that is not compressed), when no energy is lost as heat due to viscosity.

In other words: the equation applies only in a situation that is never found in practice. In a real water pipe, heat is generated by viscous drag and some linear kinetic energy is converted to rotational energy in turbulent eddies. In the ducts of an air conditioning system there is always some kinetic energy loss due to viscosity and turbulence, and air is compressible.

Physics is the art of approximation. A friction force is not strictly independent of velocity, a floor is never quite uniform, flat, or level, normal fluids (air and water) have viscosity, and dry leaves fall in air in turbulent flow. Students learn to accept that observations made in demonstrations and labs involve uncertainties.

The use of equation 1 is no exception. In real situations when pipes are wide and smooth, flow is at low velocities, and viscosity is low, Bernoulli's equation gives values that are correct to within ±10%. At higher velocities, and/or when pipes are narrow, rough, and have sharp bends, the principle still holds but the Bernoulli equation provides only a first approximation.

Example 1: a non-uniform pipe

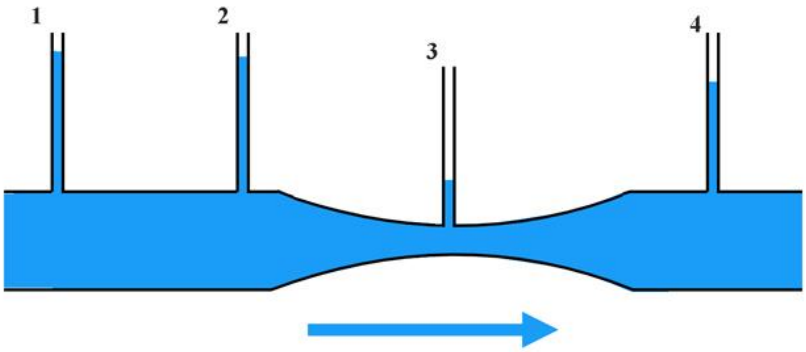

Fig 2 - water flows to the right without turbulence but with some drag due to viscosity.

Because the pipe is horizontal, $y_1 = y_2$ and equation 1 becomes …

$$P_2 = P_1 + ½\rho (v_1^2 – v_2^2) \qquad … [2]$$

1 The narrow vertical tubes in figure 2 are called *pressure taps*. The gauge pressure P_2 (in cm of water) is a little lower than P_1 because of viscous drag.

2 Because velocity is increased at 3 the pressure P_3 is lower there than that it would have been due to viscous drag if the pipe were of uniform cross section.

Note: Bernoulli's principle has very general application, even when there is turbulence and the fluid is compressible. *Seal the doors and windows of a flat roofed building to keep the pressure inside close to one atmosphere. Watch the roof lift off as the building seems to explode in a hurricane.*

Example 2: water flowing from a tank

A tank of water (figure 3) has a small hole a distance *h* below the water surface. The velocity of the upper water surface as the tank empties (v_1 in equation 1) is close to zero. The pressure is one atmosphere everywhere so $P_1 = P_2$. The difference in pressure due to height difference (y_1 - y_2) is ρgh.

Equation 1 becomes …

$$½\rho v_2^2 = \rho gh$$

and …

$$v = \sqrt{(2gh)} \qquad … [3]$$

This is *Toricelli's equation* that gives the velocity of ejection of water leaving a hole in a tank. The velocity is the same as the velocity of an object that has fallen a distance *h* in a gravitational field *g*. A moment's thought will show that this must be the case: equation 1 is a statement of the conservation of energy.

An ideal case

The diagram below shows an open tank of water with a small hole, a distance h below the water surface and a distance h above a flat table. The water stream follows a parabola with constant horizontal velocity and acceleration $-g$ in the vertical direction.

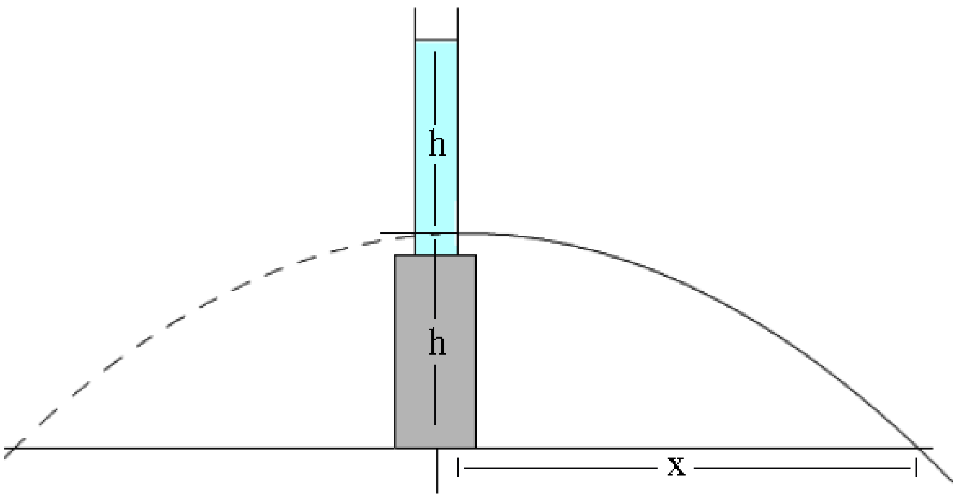

Fig 3 - water flows from a small hole near the base of an open tank.

1 The velocity of ejection is $\sqrt{(2gh)}$ (from equation 3). This is the horizontal velocity of a parabolic motion and is constant.

2 As the water stream moves from $(0, h)$ to $(x, 0)$ the water falls a distance h and the vertical velocity increases to - $\sqrt{(2gh)}$ (from the conservation of energy).

3 At the point $(x, 0)$ the horizontal and vertical water velocities have the same magnitudes and the angle that the water path intersects the horizontal is 45º.

4 The distance x is half the range (R) of a projectile launched from $(-x, 0)$.

$$R = 2u_x u_y/g$$

$$= 2\sqrt{(2gh)}\sqrt{(2gh)}\ g$$

$= 4h$ … and the distance x on the diagram is $2h$.

Note: in a real case: when the water tank is a cylinder 10 cm in diameter and 30 cm high with a wall 3 mm thick and the hole is 3 mm in diameter, viscosity and turbulence reduce the ejection velocity slightly and air resistance further reduces the range. The intersection angle is close to 47º and the range is reduced by 3-5%.