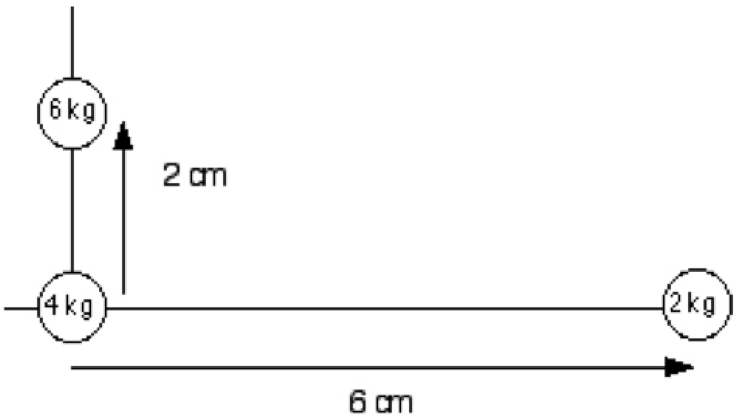
Centre of mass

For a collection of point masses on a line the distance of the centre of mass from a defined origin is given by …

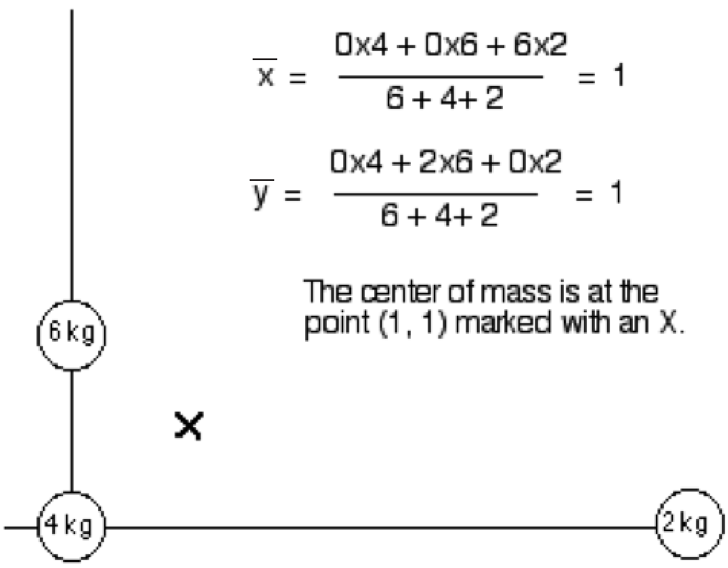
$$\bar{x} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + \ldots}{m_1 + m_2 + m_3 + \ldots}$$

The relationship is extended to two and three dimensions by finding the coordinates separately (one at a time).

In two dimensions, *selecting axes carefully to reduce calculation* …



The point-mass approximation follows …



Differentiating with respect to time gives the velocity of the centre of mass as …

$$\bar{v} = \frac{m_1v_1 + m_2v_2 + m_3v_3 + \dots}{m_1 + m_2 + m_3 + \dots}$$

Differentiating again gives the acceleration as …

$$\bar{a} = \frac{m_1a_1 + m_2a_2 + m_3a_3 + \dots}{m_1 + m_2 + m_3 + \dots}$$

If a body is continuous, the top and bottom lines become infinite sums over elements of mass *dm*. Analytical solutions may be found by integration for regular bodies like uniform triangles and hemispheres. For rigid bodies of irregular shape the centre of mass can be found by practical means. As an example consider the collection of small equal masses *m* embedded in a polystyrene sheet, (made for demonstration purposes).

A good approximation to the centre of mass (that lies inside the small circle) can be found by defining axes, measuring and calculating as above. If a correction for the mass of the polystyrene is required the centre of mass of the irregular sheet is best found by removing the lead, hanging from different points and drawing verticals, or by finding the point of balance when horizontal, or by finding the rotation axis about which no vibration occurs.

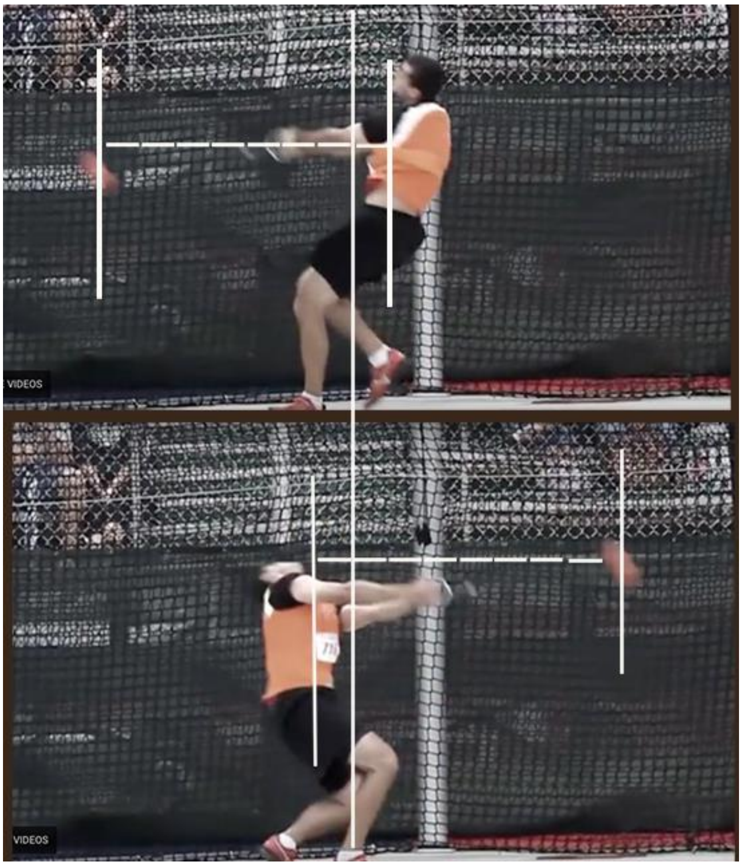
Rotation of two body systems

The hammer throw

An Olympic hammer is a ball on a chain with a mass of 7.26 kg. It is swung at spped and released to fly as far as possible. The man and the ball would rotate about their centre of mass if both were rigid bodies and clear of the ground.

In a real case https://www.youtube.com/watch?v=vAdmOrwOh7E that is an approximation, because the man in the clip is moving right to left. The ball has tangential acceleration and he is not clear of the ground and is not a rigid body. A calculation based on simple approximations with give an order of magnitude result for his mass but will not be accurate to the nearest kg.

Two frames 180 degrees apart



Marking a centre line (neglecting translation left to right) and marking two verticals that pass somewhere close to the man's centre of mass shows that the ratio of lengths is 1: 7 or 8. The apparent mass of the man is 50-60 kg. He appears to be closer to 100 kg and the value is low, but is satisfactory as a first approximation.

In principle, we see that the thrower and the hammer rotate about their mutual centre of mass.

The solar system

For the same reason the earth wobbles in orbit about the sun due to the influence of the moon, and the sun itself wobbles on a line in space because of the influence of the planets (mostly Jupiter). Perturbations in the straight line motion of a star may indicate the presence of a dark companion star, or if slight, of a large planet or two

Data for the earth, the moon, Jupiter and the sun are listed below.

Earth: mass = 5.97×10^{24} kg … radius = 6.4×10^{6} m

Moon: mass = 7.35×10^{22} kg … orbital radius = 3.84×10^{8} m.

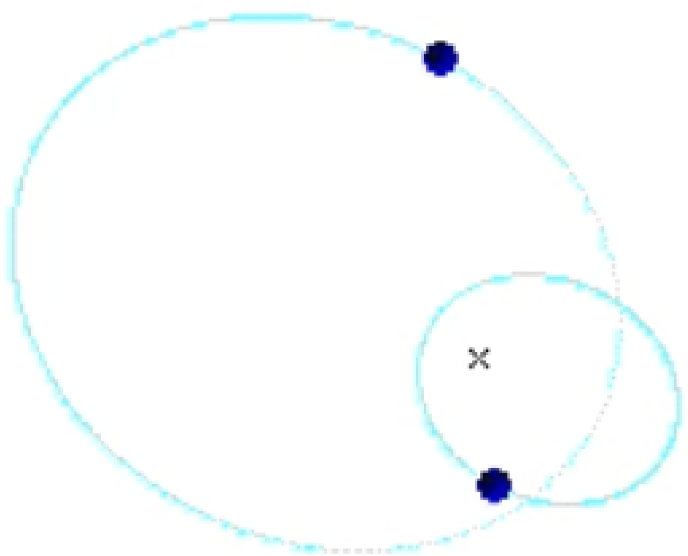
Jupiter: mass = 1.90×10^{27} kg … orbital radius = 7.80×10^{11} m.

Sun: mass = 2 00 $\times 10^{20}$ kg … radius = 9.65×10^{8} m.

It is left as an exercise to show that the centre of mass of the earth-moon system lies below the surface of the earth and that for the sun-planet system the situation is similar, with the centre of mass somewhere close to the surface of the sun.

Stars

Two stars of unequal mass are shown orbiting their centre of mass in a diagram taken from a computer generated simulation.



The centre of mass is stationary in the frame of reference and the orbits are elliptical.