Error propagation

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Errors in X and Y are expressed as $\pm\Delta x$ and $\pm\Delta y$. When X and Y are added (or subtracted), *intuitively*, as an upper bound, we have …

$$(X\pm\Delta x) + (Y\pm\Delta y) = (X+Y) \pm (\Delta x+\Delta y)$$

… the error in (X+Y) and in (X-Y) is the sum of the individual errors.

When beginning a study of physics it is important that students do not underestimate errors and claim relationships that have not been shown to be correct. The sum of the errors is an upper bound to the possible error in the sum and is often taken as sufficiently accurate. As we shall shortly discuss this is an approximation.

When X and Y are multiplied we have, as an upper bound …

$$XY + \Delta xy = (X \pm \Delta x)(Y \pm \Delta y)$$

$$= XY + X\Delta y + Y\Delta x + \Delta x\Delta y$$

Since Δx and Δy are small we can neglect $\Delta x\Delta y$ and we have …

$$\Delta xy = X\Delta y + Y\Delta x$$

Dividing by XY we have …

$$\Delta xy/XY = \Delta x/X + \Delta y/Y$$

and $$\Delta xy = XY(\Delta x/X + \Delta y/Y)$$

…. the fractional error in XY is given (as an upper bound) by adding the fractional errors in X and Y.

Because (as above) when beginning a study of physics it is important that students do not underestimate errors this upper bound to the possible error in the product is often taken as sufficiently accurate. Since division is multiplication by a negative power the relationship holds also for quotients and for powers.

Suppose $X = 36 \pm 6$ and $Y = 64 \pm 7$

When adding …. $X + Y = 100 \pm 10$

… rounding the error (13) to one significant figure.

When multiplying … $\Delta xy = XY(\Delta x/X + \Delta y/Y) = 36x64(6/36 + 7/64)$

$XY = 2300 \pm 600$… rounding the error to one significant figure.

A deeper understanding

The weakness when using upper bounds as errors in sums and products is the statistical nature of the stated errors. If a standard deviation is used then on average two out of three values will lie closer to the mean than the stated error. To take account of this when error distributions are Gaussian (*Normal*) it can be shown that the errors and fractional errors are properly added in quadrature.

Addition in quadrature

When adding X and Y … $(\Delta x+\Delta y)$ becomes $\sqrt{(\Delta x^2+\Delta y^2)}$

When multiplying … $(\Delta x/X + \Delta y/Y)$ becomes $\sqrt{[(\Delta x/X)^2 + (\Delta y/Y)^2]}$

These relationps hips they may be remembered and used when appropriate.

Suppose X = 36 ± 6 and Y = 64 ± 7

$$\Delta(x+y) = \sqrt{(\Delta x^2+\Delta y^2)}$$

$$= \sqrt{(6^2+\Delta 7^2)} = 9.2$$

$$X + Y = 100 \pm 9$$

Note: 9 is ~30% less than 13 but because of rounding, the two calculated errors appear to be almost the same.

$$\Delta(xy) = XY\sqrt{[(\Delta x/X)^2 + (\Delta y/Y)^2]}$$

$$= 36x64\sqrt{[(6/36)^2 + (764)^2]} = 459$$

$$XY = 2300 \pm 500$$

Again, because of rounding the calculated errors to one significant figure, the difference between 459 and 636, which is close to 30%, is not so obvious in the result.

Powers

Note: because addition in quadrature is derived for a statistical process it is important that the two values and their errors be *independent*. When calculating the error in a power (for example in X^2) it is important to remember that addition in quadrature does not apply.

$$\Delta(x^2) = X^2 (\Delta x/X + (\Delta x/X)$$

Suppose X = 36 ± 6

$$\Delta(x^2) = 2x36^2 (6/36) = 432$$

$$X = 1300 \pm 400$$